Photon-emission rate from atomic systems in the CSL model

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Abstract
We calculate the photon-emission rate from a general atomic system in the mass-proportional continuous spontaneous localization (CSL) model. For an isolated charged particle emitting kilovolt gamma rays, our results agree with those obtained by Fu. For a neutral atomic system, photon emission is strongly suppressed for photon wavelengths much larger than the atomic radius. However, for kilovolt gamma rays, Fu's result is modified by a structure factor that is of order unity, giving no rate suppression. Our calculation is readily generalized to the case of non-white noise, noise couplings that are not mass-proportional, and general (non-Gaussian) spatial correlation functions, and corresponding results are given. We briefly discuss the implications of our calculation for upper bounds on the CSL model parameters.

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1. Introduction

Stochastic modifications of the Schrödinger equation, such as the continuous spontaneous localization (CSL) model, solve the measurement problem in quantum theory by giving an objective account of state vector reduction [1]. To assess the viability of these models, it is necessary to estimate lower and upper bounds on the stochastic model parameters, as surveyed in a recent paper of Adler [2]. An important upper bound on the stochastic rate parameter comes from a calculation by Fu [3] of the rate of noise-induced gamma radiation from free electrons, which he compares with the observed bound on 11 kV gamma radiation from germanium. Adler suggested in [2] that in a neutral atomic system, radiation from protons, in the case of mass-proportional noise couplings, will largely cancel the radiation from electrons. Our aim in this paper is to check this assertion by a detailed calculation of stochastic noise-induced radiation in a general atomic system. We find that the asserted cancellation is present only for very long wavelength photons, whereas for the 11 kV gamma rays figuring in Fu's bound, the radiation from protons somewhat enhances, rather than reducing, that from electrons. This

result can be simply understood as the effect of inclusion of the space coordinate-dependent phase factor for the radiated wave.

Thus for white noise, the upper bound on the CSL rate parameter is six orders of magnitude lower than estimated in [2], and hence is three orders of magnitude smaller than the lower bounds estimated in [2] from processes of latent image formation, assuming that latent image formation (and not subsequent development) corresponds to state vector reduction. Hence if the assumptions on which these lower bounds are based are correct, the white-noise CSL model is disfavored. White noise is of course an idealization, and our calculation can be readily extended to the case of non-white noise. For non-white noise with a spectral cutoff below 11 kV, there is no 11 kV gamma radiation, and so in this case the germanium experiment does not set a bound on the CSL model rate parameter, and there is no conflict with the lower bounds estimated in [2].

This paper is organized as follows. In section 2 we outline the basic strategy of the calculation, which is to replace the real noise of the CSL model by an imaginary noise, that can be represented by a perturbation term in the Hamiltonian. We write down the general form of the Hamiltonian, and give the noise structure in the white-noise and non-white-noise cases.

In section 3 we use standard atomic physics methods [4] to derive a master formula for the noise-induced photon radiation rate, in both the white-noise and the non-white-noise cases. In section 4 we evaluate this formula for a single free electron, recovering the result of Fu [3] when his approximations are made. In section 5 we evaluate the master formula for a hydrogenic atom, and in section 6 for a general atomic system. In section 7 we state the generalization of our results to a noise perturbation with general (not necessarily mass proportional) couplings to the particles, and with general spatial and time correlation functions. We conclude with a brief discussion of the implications of our calculation for CSL model phenomenology.

2. General strategy, Hamiltonian and noise structure

In the CSL model, the stochastic Schrödinger equation obeyed by the wavefunction ψ takes the form $d\psi = -(i/\hbar)H\psi\, dt + \mathcal{N}\psi + \cdots$, with H the usual Hamiltonian, with the noise term $\mathcal{N}$ real valued, and with the ellipsis . . . representing additional nonlinear terms needed to preserve state vector normalization. A real-valued choice for the noise term corresponds to an imaginary addition to the Hamiltonian, and is necessary to obtain a model that describes state vector reduction. An alternative stochastic Schrödinger equation can be written with an imaginary noise term, which does not require additional nonlinear terms in the Schrödinger equation for norm preservation. This Schrödinger evolution does not lead to state vector reduction, but for the case of white noise, it is a well-known result that the noise average of the density matrix obeys the same evolution equation in the real and imaginary noise cases. Since the mean rate for noise-induced transitions can be calculated from the noise averaged density matrix, this implies that one can use the imaginary noise equation to calculate the mean rate for such transitions. Hence, to leading order, one can represent the noise perturbation as a self-adjoint perturbation on the Hamiltonian H, and use standard second-order perturbation theory to evaluate its effects.

The usual justification for the use of imaginary noise is based on a calculation of the density matrix evolution in the real and imaginary noise cases using the Itô calculus, which as already noted, assumes white noise. Adler and Bassi [5] have recently shown, however, that in the case of non-white Gaussian noise, the noise-averaged density matrix evolutions are still the same for the real and imaginary noise cases, through second order in the noise term. Hence, in the second-order perturbation calculations of this paper, we can use an imaginary noise term to calculate the effects of non-white noise as well as white noise.

We will thus be considering a Hamiltonian of the form

$$H = H_0 + H_{\text{em}} + H_n, \tag{1}$$

with H_0 being the atomic system Hamiltonian, H_{em} the electromagnetic perturbation describing photon emission and H_n the perturbation describing the noise. For a system of N particles of charges e_j and masses m_j, the electromagnetic perturbation is

$$H_{\text{em}} = \sum_{j=1}^{N} \frac{\mathrm{i} e_j \hbar}{m_j c} \vec{A}(\vec{x}_j) \cdot \vec{\nabla}_{x_j} + O(\vec{A}^2), \tag{2}$$

with the electromagnetic potential, for field quantization in a cubical box of size L, given by

$$\vec{A}(\vec{x}) = \sum_{\vec{p}} \sqrt{\frac{2\pi\hbar c^2}{\omega_p L^3}} \left[a_p \vec{\epsilon}_p \, \mathrm{e}^{\mathrm{i}(\vec{p}\cdot\vec{x} - \omega_p t)} + a_p^\dagger \vec{\epsilon}_p \, \mathrm{e}^{-\mathrm{i}(\vec{p}\cdot\vec{x} - \omega_p t)} \right], \tag{3}$$

where $\omega_p = pc$, and where the numerical value of a unit unrationalized charge e is $e^2/(\hbar c) \simeq 1/137.04$. Since we are only interested in the matrix element for emitting a single photon of wave number $\vec{p}$, we pull this term out from equation (3) and, separating off the time dependence, write the electromagnetic perturbation as

$$H_{\text{em}} = \mathrm{e}^{\mathrm{i}\omega_p t} \mathcal{W}^p(\{\vec{x}\}),$$
$$\mathcal{W}^p(\{\vec{x}\}) = a_p^\dagger \sqrt{\frac{2\pi\hbar c^2}{\omega_p L^3}} \sum_j \frac{\mathrm{i} e_j \hbar}{m_j c} \, \mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{x}_j} \vec{\epsilon}_p \cdot \vec{\nabla}_j, \tag{4}$$

where $\vec{\nabla}_j$ is an abbreviation for $\vec{\nabla}_{x_j}$.

In the CSL model with mass-proportional couplings, the noise perturbation can be written as

$$H_n = \int \mathrm{d}^3 z \, \frac{\mathrm{d} W_t(\vec{z})}{\mathrm{d} t} \mathcal{V}(\vec{z}, \{x\}), \qquad \mathcal{V}(\vec{z}, \{\vec{x}\}) = -\frac{\hbar}{m_N} \sum_j m_j g(\vec{z} - \vec{x}_j). \tag{5}$$

Here $g(\vec{x})$ is a spatial correlation function, conventionally taken as the Gaussian

$$g(\vec{x}) = \left(\frac{\alpha}{2\pi} \right)^{3/2} \mathrm{e}^{-\alpha\vec{x}^2/2} = (\sqrt{2\pi} r_c)^{-3} \, \mathrm{e}^{-\vec{x}^2/2r_c^2}, \tag{6}$$

with $\alpha^{-\frac{1}{2}} = r_c$, and with r_c conventionally taken as 10^{-5} cm. In the case of white noise, $\mathrm{d} W_t$ is an Itô calculus differential that obeys

$$\mathrm{d} W_t(\vec{x}) \, \mathrm{d} W_t(\vec{y}) = \gamma \, \mathrm{d} t \, \delta^3(\vec{x} - \vec{y}), \tag{7}$$

with γ being the noise strength parameter. The corresponding formula for the case of non-white noise is

$$E\left[\frac{\mathrm{d} W_t(\vec{x})}{\mathrm{d} t} \frac{\mathrm{d} W_{t'}(\vec{y})}{\mathrm{d} t'} \right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \gamma(\omega) \, \mathrm{e}^{-\mathrm{i}\omega(t-t')} \delta^3(\vec{x} - \vec{y}), \tag{8}$$

with $E[\cdots]$ denoting the expectation or average over the noise. When $\gamma(\omega)$ is a constant γ, equation (8) reduces, on integration over t', to equation (7).

3. Master equation for the radiation rate

According to equations (4) and (5), the total perturbation on the atomic Hamiltonian H_0 is

$$V(t) = \int d^3z \frac{dW_t(\vec{z})}{dt} \mathcal{V}(\vec{z}, \{\vec{x}\}) + e^{i\omega_p t} \mathcal{W}^p(\{\vec{x}\}). \tag{9}$$

Expanding the transition amplitude in a perturbation series following the methods of [4], we get

$$\langle f|U_I(t,0)|i\rangle = 1 + \mathcal{T}_{fi}^{(1)} + \mathcal{T}_{fi}^{(2)} + \cdots,$$

$$\mathcal{T}_{fi}^{(2)} = -\frac{1}{\hbar^2} \int_0^t ds \int_0^s du \sum_k \langle f|V_I(s)|k\rangle \langle k|V_I(u)|i\rangle$$

$$= -\frac{i}{2\pi\hbar^2} \int_0^t ds \int_0^t du \int_{-\infty}^{\infty} dE \, e^{\frac{i}{\hbar}(E_f-E)s} e^{\frac{i}{\hbar}(E-E_i)u} \sum_k \frac{V_{fk}(s)V_{ki}(u)}{E_i + i\eta - E_k}, \tag{10}$$

where in the first line of the formula for $\mathcal{T}_{fi}^{(2)}$, V_I denotes the interaction picture perturbation, and in the second line V_{fk} and V_{ki} denote matrix elements of the Schrödinger picture perturbation. To calculate the noise-induced radiation, we are only interested in the terms in equation (10) that are bilinear in the electromagnetic and noise perturbations, so on substituting equation (9) and dropping irrelevant terms, we get

$$\mathcal{T}_{fi}^{(2)} = \frac{-i}{2\pi\hbar^2} \int_{-\infty}^{\infty} dE$$

$$\times \left(\int_0^t ds \, e^{\frac{i}{\hbar}(E_f-E)s} \int d^3z \frac{dW_s(\vec{z})}{ds} \int_0^t du \, e^{\frac{i}{\hbar}(E-E_i+\hbar\omega_p)u} \sum_k \frac{\mathcal{V}_{fk}(\vec{z})\mathcal{W}_{ki}^p}{E + i\eta - E_k} \right.$$

$$\left. + \int_0^t ds \, e^{\frac{i}{\hbar}(E_f-E+\hbar\omega_p)s} \int_0^t du \, e^{\frac{i}{\hbar}(E-E_i)u} \int d^3z \frac{dW_u(\vec{z})}{du} \sum_k \frac{\mathcal{W}_{fk}^p \mathcal{V}_{ki}(\vec{z})}{E + i\eta - E_k} \right). \tag{11}$$

Taking the squared modulus of equation (11), averaging over the noise, and using the formulas for representations of the Dirac delta function given in [4], in the large time limit we obtain in the white-noise case

$$E\left[|\mathcal{T}_{fi}^{(2)}|^2\right] = \frac{\gamma t}{\hbar^2} \int d^3z \left| \sum_k \frac{\mathcal{V}_{fk}(\vec{z})\mathcal{W}_{ki}^p}{E_i - \hbar\omega_p + i\eta - E_k} + \frac{\mathcal{W}_{fk}^p \mathcal{V}_{ki}(\vec{z})}{E_f + \hbar\omega_p + i\eta - E_k} \right|^2, \tag{12}$$

with the corresponding equation in the non-white-noise case taking the form

$$E\left[|\mathcal{T}_{fi}^{(2)}|^2\right] = \frac{t}{\hbar^2} \gamma \left(\omega_p + \frac{E_f - E_i}{\hbar} \right)$$

$$\times \int d^3z \left| \sum_k \frac{\mathcal{V}_{fk}(\vec{z})\mathcal{W}_{ki}^p}{E_i - \hbar\omega_p + i\eta - E_k} + \frac{\mathcal{W}_{fk}^p \mathcal{V}_{ki}(\vec{z})}{E_f + \hbar\omega_p + i\eta - E_k} \right|^2. \tag{13}$$

Equations (12) and (13) are the master equations from which we shall calculate the noise-induced radiation rate, by substituting the matrix elements of $\mathcal{V}$ and $\mathcal{W}^p$ appropriate to the various cases of interest.

4. Free electron: repeating Fu's calculation

As a first application of equation (12), and a check, let us repeat the calculation of Fu [3] for the case of a single free electron. Assuming that the electron is initially at rest, the initial, final and intermediate state electron wavefunctions are

$$\psi_i = \frac{1}{\sqrt{L^3}}, \qquad \psi_f = \frac{e^{i\vec{q}\cdot\vec{x}}}{\sqrt{L^3}}, \qquad \psi_k = \frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{L^3}}. \tag{14}$$

From equations (4) and (5), as specialized to a single particle of charge e (with $e^2/(\hbar c) \simeq 1/137$) and mass m, the needed matrix elements are

$$\mathcal{W}^p_{ki} = 0, \qquad \mathcal{W}^p_{fk} = -\sqrt{\frac{2\pi\hbar c}{pL^3}} \frac{e\hbar}{mc} \vec{\epsilon}_p \cdot \vec{q} \delta_{\vec{k}-\vec{p}-\vec{q}}, \tag{15}$$

and

$$\mathcal{V}_{ki}(z) = -\frac{\hbar m}{m_N L^3} e^{-i\vec{k}\cdot\vec{z}-\frac{1}{2}\vec{k}^2 r_c^2}, \qquad \mathcal{V}_{fk}(z) = -\frac{\hbar m}{m_N L^3} e^{i(\vec{k}-\vec{q})\cdot\vec{z}-\frac{1}{2}(\vec{k}-\vec{q})^2 r_c^2}. \tag{16}$$

Substituting these into equation (12), we get for the noise-averaged squared matrix element

$$E\big[|\mathcal{T}^{(2)}_{fi}|^2\big] = \frac{\gamma t}{\hbar^2} \int d^3z \left| \frac{\hbar}{m_N L^3} \sqrt{\frac{2\pi\hbar c}{pL^3}} \frac{e\hbar}{c} \vec{\epsilon}_p \cdot \vec{q} \frac{e^{-i(\vec{p}+\vec{q})\cdot\vec{z}-\frac{1}{2}(\vec{p}+\vec{q})^2 r_c^2}}{-\frac{\hbar^2}{2m}(p^2+2\vec{p}\cdot\vec{q})+\hbar cp+i\eta} \right|^2. \tag{17}$$

Fu notes that when the photon momentum p is much larger than the inverse correlation length $1/r_c$, the Gaussian factor in equation (17) forces the electron and photon to emerge nearly back to back, that is, $\vec{q} \simeq -\vec{p}$. As a result

$$\frac{\hbar cp}{-\frac{\hbar^2}{2m}(p^2+2\vec{p}\cdot\vec{q})} \simeq \frac{2mc^2}{\hbar pc}, \tag{18}$$

which is of order 100 for $\hbar pc = 11$ keV. Thus one can to a good approximation keep only the term $\hbar cp$ in the denominator of equation (17), which then simplifies to

$$E\big[|\mathcal{T}^{(2)}_{fi}|^2\big] = \frac{\gamma t}{\hbar^2} \left(\frac{\hbar}{m_N L^3}\right)^2 \frac{2\pi\hbar c}{p} \frac{e^2}{c^4 p^2} (\vec{\epsilon}_p \cdot \vec{q})^2 e^{-(\vec{p}+\vec{q})^2 r_c^2}. \tag{19}$$

Integrating over phase space for the electron and photon, summing over photon polarizations and dividing by the elapsed time, we get for the radiated power per unit photon momentum space volume and per unit time

$$\frac{dP}{d^3 p} = \left(\frac{L}{2\pi}\right)^6 \int d^3q \sum_\epsilon E\big[|\mathcal{T}^{(2)}_{fi}|^2\big] \frac{1}{t}. \tag{20}$$

Carrying out the integrals and polarization sum, and replacing the noise parameter γ by a new parameter λ defined by $\gamma = 8\pi^{3/2} r_c^3 \lambda$, we get finally for the power radiation rate

$$\frac{dP}{dp} = \frac{\hbar}{c^3} \frac{e^2 \lambda}{\pi r_c^2 m_N^2 p}. \tag{21}$$

This is in agreement with the result obtained by Fu [3], when our unrationalized charge squared e^2 is replaced by $e^2/(4\pi)$, corresponding to Fu's use of a rationalized charge convention.

5. Hydrogenic atom

We consider next a hydrogenic atom, with oppositely charged particles of masses m_1 and m_2. Equation (5) for $\mathcal{V}(\vec{z}, \{\vec{x}\})$ now takes the form

$$\mathcal{V}(\vec{z}, \{\vec{x}\}) = -\frac{\hbar}{m_N}\mathcal{M}(\vec{z}, \{\vec{x}\}), \tag{22}$$

$$\mathcal{M}(\vec{z}, \{\vec{x}\}) = m_1 g(\vec{z} - \vec{x}_1) + m_2 g(\vec{z} - \vec{x}_2). \tag{23}$$

Introducing the center-of-mass coordinate $\vec{X}$, total mass M, relative coordinate $\vec{x}$ and reduced mass μ, by

$$\vec{X} = \frac{m_1}{M}\vec{x}_1 + \frac{m_2}{M}\vec{x}_2, \qquad \vec{x} = \vec{x}_1 - \vec{x}_2, \tag{24}$$

$$M = m_1 + m_2, \qquad \mu = \frac{m_1 m_2}{M}, \tag{25}$$

we can use the fact that the Bohr radius a_0 is much smaller than r_c to approximate $\mathcal{M}(\vec{z}, \{\vec{x}\})$ as follows:

$$\begin{aligned}
\mathcal{M}(\vec{z}, \{\vec{x}\}) &= m_1 g(\vec{z} - \vec{x}_1) + m_2 g(\vec{z} - \vec{x}_2) \\
&= M g(\vec{z} - \vec{X}) + \frac{m_1 m_2}{2M}(\vec{x} \cdot \vec{\nabla}_z)^2 g(\vec{z} - \vec{X}) + \\
&\cong M g(\vec{z} - \vec{X}),
\end{aligned} \tag{26}$$

giving

$$\mathcal{W}^p(\{\vec{x}\}) = a_p^\dagger \sqrt{\frac{2\pi\hbar c}{pL^3}}\frac{ie\hbar}{c}\vec{\epsilon}_p \cdot \left(\frac{1}{m_1}\mathrm{e}^{-i\vec{p}\cdot\vec{x}_1}\vec{\nabla}_1 - \frac{1}{m_2}\mathrm{e}^{-i\vec{p}\cdot\vec{x}_2}\vec{\nabla}_2\right), \tag{27}$$

$$\mathcal{V}(\vec{z}, \{\vec{x}\}) \cong -\frac{\hbar M}{m_N}g(\vec{z} - \vec{X}). \tag{28}$$

The initial, final and intermediate state atomic wavefunctions are now

$$\psi_i = \frac{1}{\sqrt{L^3}}u_{\hat{i}}(\vec{x}), \qquad \psi_f = \frac{\mathrm{e}^{i\vec{q}\cdot\vec{X}}}{\sqrt{L^3}}u_{\hat{f}}(\vec{x}), \qquad \psi_k = \frac{\mathrm{e}^{i\vec{k}\cdot\vec{X}}}{\sqrt{L^3}}u_{\hat{k}}(\vec{x}), \tag{29}$$

where we use carets to denote the labels of hydrogenic internal states. Defining

$$\mathcal{O}(\vec{k}) = \frac{i}{M}\left(\mathrm{e}^{-i\frac{m_2}{M}\vec{p}\cdot\vec{x}} - \mathrm{e}^{i\frac{m_1}{M}\vec{p}\cdot\vec{x}}\right)\vec{\epsilon}_p \cdot \vec{k} + \left(\frac{1}{m_1}\mathrm{e}^{-i\frac{m_2}{M}\vec{p}\cdot\vec{x}} + \frac{1}{m_2}\mathrm{e}^{i\frac{m_1}{M}\vec{p}\cdot\vec{x}}\right)\vec{\epsilon}_p \cdot \vec{\nabla}_x, \tag{30}$$

we find that the matrix elements entering the master formula are

$$\mathcal{W}_{ki}^p = \sqrt{\frac{2\pi\hbar c}{pL^3}}\frac{ie\hbar}{c}\langle\hat{k}|\mathcal{O}(\vec{0})|\hat{i}\rangle\delta_{\vec{k}+\vec{p}} \tag{31}$$

$$\mathcal{W}_{fk}^p = \sqrt{\frac{2\pi\hbar c}{pL^3}}\frac{ie\hbar}{c}\langle\hat{f}|\mathcal{O}(\vec{k})|\hat{k}\rangle\delta_{\vec{k}-\vec{p}-\vec{q}}, \tag{32}$$

and

$$\mathcal{V}_{ki} = -\frac{\hbar M}{m_N L^3} \, e^{-i\vec{k}\cdot\vec{z} - \frac{1}{2}\vec{k}^2 r_c^2} \delta_{\hat{k}\hat{\imath}} \tag{33}$$

$$\mathcal{V}_{fk} = -\frac{\hbar M}{m_N L^3} \, e^{i(\vec{k}-\vec{q})\cdot\vec{z} - \frac{1}{2}(\vec{k}-\vec{q})^2 r_c^2} \delta_{\hat{f}\hat{k}}. \tag{34}$$

Then without any further approximation we find

$$E\big[|\mathcal{T}_{fi}^{(2)}|^2\big] = \frac{\gamma t}{\hbar^2} \left(\frac{\hbar M}{m_N L^3}\right)^2 \frac{2\pi\hbar c}{p} \frac{e^2\hbar^2}{c^2} \, e^{-(\vec{p}+\vec{q})^2 r_c^2}$$

$$\times \left| -\frac{\langle\hat{f}|\mathcal{O}(\vec{0})|\hat{\imath}\rangle}{E_{fi} + \frac{\hbar^2 p^2}{2M} + \hbar c p - i\eta} + \frac{\langle\hat{f}|\mathcal{O}(\vec{p}+\vec{q})|\hat{\imath}\rangle}{E_{fi} + \frac{\hbar^2 q^2}{2M} + \hbar c p - \frac{\hbar^2(\vec{p}+\vec{q})^2}{2M} + i\eta} \right|^2, \tag{35}$$

with $E_{fi} \equiv E_{\hat{f}} - E_{\hat{\imath}}$ being the internal energy difference between the final and initial atomic states. The radiated power, per unit photon momentum space volume and per unit time, now requires a sum over the final internal atomic state $\hat{f}$, and is given by

$$\frac{\mathrm{d}P}{\mathrm{d}^3 p} = \left(\frac{L}{2\pi}\right)^6 \int \mathrm{d}^3 q \sum_{\hat{f},\epsilon} E\big[|\mathcal{T}_{fi}^{(2)}|^2\big] \frac{1}{t}. \tag{36}$$

Note that when $\vec{p} + \vec{q} = 0$, the two terms in equation (37) cancel. Since the Gaussian $e^{-(\vec{p}+\vec{q})^2 r_c^2}$ constrains $|\vec{p}+\vec{q}|$ to be not much larger than $1/r_c$, we can make this cancellation explicit by expanding in the small parameter

$$\frac{\hbar^2 \vec{p}\cdot(\vec{p}+\vec{q})}{M\big(\hbar c p + \frac{\hbar^2 p^2}{2M} + E_{fi}\big)} \equiv \frac{\hbar^2 \vec{p}\cdot(\vec{p}+\vec{q})}{M D_0}, \tag{37}$$

which keeping the leading two terms, and writing $\vec{p}\cdot\vec{x} = pz$, gives

$$\frac{\mathrm{d}P}{\mathrm{d}p} = p\frac{\hbar^3}{c}\left(\frac{M}{m_N}\right)^2 \frac{e^2\lambda}{\pi r_c^2} \sum_{\hat{f}} \left\{ \frac{1}{M^2 D_0^2} \big|\langle\hat{f}| e^{-i\frac{m_2}{M}pz} - e^{i\frac{m_1}{M}pz}|\hat{\imath}\rangle\big|^2 \right.$$

$$\left. + \frac{p^2\hbar^4}{M^2 D_0^4} \left|\langle\hat{f}|\left(\frac{1}{m_1}e^{-i\frac{m_2}{M}pz} + \frac{1}{m_2}e^{i\frac{m_1}{M}pz}\right)\frac{\partial}{\partial x}|\hat{\imath}\rangle\right|^2 \right\}. \tag{38}$$

For small p, this expression can be further simplified to

$$\frac{\mathrm{d}P}{\mathrm{d}p} = p^3\frac{\hbar^3}{c}\left(\frac{M}{m_N}\right)^2 \frac{e^2\lambda}{\pi r_c^2} \sum_{\hat{f}} \left\{ \frac{1}{M^2 E_{fi}^2}|\langle\hat{f}|z|\hat{\imath}\rangle|^2 + \frac{\hbar^4}{M^2 E_{fi}^4 \mu^2}\left|\langle\hat{f}|\frac{\partial}{\partial x}|\hat{\imath}\rangle\right|^2 \right\}$$

$$= 2p^3\frac{\hbar^3}{c}\frac{1}{m_N^2}\frac{e^2\lambda}{\pi r_c^2} \sum_{\hat{f}} \frac{|\langle\hat{f}|z|\hat{\imath}\rangle|^2}{E_{fi}^2}, \tag{39}$$

where we have used the dipole approximation formula

$$\left|\langle\hat{f}|\frac{\partial}{\partial x}|\hat{\imath}\rangle\right| = \frac{\mu E_{fi}}{\hbar^2}|\langle\hat{f}|x|\hat{\imath}\rangle|, \tag{40}$$

which shows that the two terms in equation (39) make equal contributions. The sum in equation (39) has been evaluated in a closed form by Dalgarno and Kingston [6], with the result

$$\sum_{\hat{f}} \frac{|\langle \hat{f}|z|\hat{i}\rangle|^2}{E_{fi}^2} = \frac{43}{8} \frac{\mu^2 a_0^6}{\hbar^4}, \tag{41}$$

giving an explicit expression for the small p radiation rate.

However, for 11 kV photons, the small p approximation does not apply, and instead we can simplify the formulas by making the approximation $D_0 \approx \hbar c p$, as was done by Fu in his calculation. The radiation rate then becomes

$$\frac{\mathrm{d}P}{\mathrm{d}p} = \frac{1}{p} \frac{\hbar}{c^3} \left(\frac{M}{m_N}\right)^2 \frac{e^2 \lambda}{\pi r_c^2} \sum_{\hat{f}} \left\{ \frac{1}{M^2} \left|\langle \hat{f}|\mathrm{e}^{-\mathrm{i}\frac{m_2}{M}pz} - \mathrm{e}^{\mathrm{i}\frac{m_1}{M}pz}|\hat{i}\rangle\right|^2 \right.$$
$$\left. + \frac{\hbar^2}{M^2 c^2} \left|\langle \hat{f}| \left(\frac{1}{m_1}\mathrm{e}^{-\mathrm{i}\frac{m_2}{M}pz} + \frac{1}{m_2}\mathrm{e}^{\mathrm{i}\frac{m_1}{M}pz}\right) \frac{\partial}{\partial x}|\hat{i}\rangle\right|^2 \right\}, \tag{42}$$

which, using the completeness of the hydrogen spectrum, can be simplified to

$$\frac{\mathrm{d}P}{\mathrm{d}p} = \frac{1}{p} \frac{\hbar}{c^3} \left(\frac{M}{m_N}\right)^2 \frac{e^2 \lambda}{\pi r_c^2} \sum_{\hat{f}} \left\{ \frac{1}{M^2} \langle \hat{i}|2 - 2\cos pz|\hat{i}\rangle \right.$$
$$\left. + \frac{\hbar^2}{M^2 c^2} \langle \hat{i}| \frac{\overleftarrow{\partial}}{\partial x} \left(\frac{1}{m_1^2} + \frac{1}{m_2^2} + \frac{2}{m_1 m_2}\cos pz\right) \frac{\partial}{\partial x}|\hat{i}\rangle \right\}. \tag{43}$$

The ratio of the second term to the first can be shown to be of order $(e^2/\hbar c)^2$, so the second term can be neglected. Evaluating the first term using the hydrogen atom ground state wavefunction, we find the final result for high p to be

$$\frac{\mathrm{d}P}{\mathrm{d}p} = 2 \left[1 - \frac{1}{\left[1 + \left(\frac{pa_0}{2}\right)^2\right]^2} \right] \frac{1}{p} \frac{\hbar}{c^3} \frac{1}{m_N^2} \frac{e^2 \lambda}{\pi r_c^2}. \tag{44}$$

For small p this expression is suppressed with respect to the rate calculated by Fu, but for large p it approaches twice Fu's rate, because when the photon wave length is much smaller than the atomic radius, the electron and proton radiation rates add incoherently. For 11 kV gamma radiation from hydrogen, the rate given by equation (44) is about 1.8 times the rate for a free electron. The structure of the first term in equations (42) and (43) can be readily understood in terms of the phase factor that appears in the formula for the radiation rate of a distributed charge system, as in equations (13)–(33) and (13)–(37) of the text of Panofsky and Phillips [7].

6. Many-body system

We turn next to a general n-particle atomic system, for which the electromagnetic and noise perturbations are given by equations (4) and (5), with the sum over j extending from 1 to n. In order to take account of overall momentum conservation, we separate the coordinates of the particles into a center-of-mass coordinate $\vec{X}$ and internal coordinates $\vec{\xi}_i$, $i = 1, \ldots, n-1$,

by writing

$$
\begin{aligned}
M &= \sum_{j=1}^{n} m_j, \\
\vec{x}_i &= \vec{\xi}_i + \vec{X} \qquad i = 1, \dots, n-1, \\
\vec{x}_n &= \vec{X} - \frac{1}{m_n} \sum_{j=1}^{n-1} m_j \vec{\xi}_j, \\
\vec{\xi}_i &= \vec{x}_i - \sum_{j=1}^{n} \frac{m_j \vec{x}_j}{M}, \\
\vec{X} &= \sum_{j=1}^{n} \frac{m_j \vec{x}_j}{M}.
\end{aligned}
\tag{45}
$$

In the following equations, $\vec{\nabla}_j$ denotes the partial derivative with respect to the original coordinate $\vec{x}_j$, not the derivative with respect to the internal coordinate $\vec{\xi}_j$. Straightforward calculations show that the commutator of this partial derivative with an internal coordinate is given by

$$
[\vec{a} \cdot \vec{\nabla}_i, \vec{b} \cdot \vec{\xi}_j] = \vec{a} \cdot \vec{b} \left(\delta_{ij} - \frac{m_i}{M} \right),
\tag{46}
$$

and also that the Jacobian J of the transformation of equation (45) is given by

$$
J = \frac{\partial(\vec{x}_1 \dots \vec{x}_n)}{\partial(\vec{X}\vec{\xi}_1 \dots \vec{\xi}_{n-1})} = (-1)^{n-1} \left(1 + \frac{\sum_{j=1}^{n-1} m_j}{m_n} \right)^3.
\tag{47}
$$

Moreover, the kinetic term of the unperturbed Hamiltonian is separated by the transformation of equation (45) into a center-of-mass part and an internal part

$$
\sum_{i=1}^{n} \frac{\vec{\nabla}_i^2}{2m_i} = \frac{\vec{\nabla}_X^2}{2M} + \sum_{i=1}^{n-1} \frac{\vec{\nabla}_{\xi_i}^2}{2m_i} - \frac{1}{2M} \left(\sum_{i=1}^{n-1} \vec{\nabla}_{\xi_i} \right)^2,
\tag{48}
$$

so that we know that wavefunctions are of the factorized form

$$
\psi_i = \frac{1}{\sqrt{L^3}} u_{\hat{\imath}}(\{\vec{\xi}\}) \qquad \psi_f = \frac{e^{i\vec{q} \cdot \vec{X}}}{\sqrt{L^3}} u_{\hat{f}}(\{\vec{\xi}\}) \qquad \psi_k = \frac{e^{i\vec{k} \cdot \vec{X}}}{\sqrt{L^3}} u_{\hat{k}}(\{\vec{\xi}\}).
\tag{49}
$$

Using the center-of-mass transformation and the factorized wavefunctions, the noise and radiation matrix elements needed for the master formula of equation (12) are calculated to be

$$
\mathcal{W}_{ki}^p = \sqrt{\frac{2\pi\hbar c}{pL^3}} \frac{i\hbar}{c} \langle \hat{k} | \sum_j e^{-i\vec{p} \cdot \vec{\xi}_j} e_j \frac{\vec{\epsilon}_p}{m_j} \cdot \vec{\nabla}_j | \hat{\imath} \rangle \delta_{\vec{k}+\vec{p}}
\tag{50}
$$

$$
\mathcal{W}_{fk}^p = \sqrt{\frac{2\pi\hbar c}{pL^3}} \frac{i\hbar}{c} \langle \hat{f} | \sum_j e^{-i\vec{p} \cdot \vec{\xi}_j} e_j \left(i\frac{\vec{\epsilon}_p \cdot \vec{k}}{M} + \frac{\vec{\epsilon}_p}{m_j} \cdot \vec{\nabla}_j \right) | \hat{k} \rangle \delta_{\vec{k}-\vec{p}-\vec{q}}
\tag{51}
$$

and

$$
\mathcal{V}_{ki}(\vec{z}) = -\frac{\hbar}{m_N L^3} e^{-i\vec{k} \cdot \vec{z} - \frac{1}{2}\vec{k}^2 r_c^2} \langle \hat{k} | \sum_j e^{i\vec{k} \cdot \vec{\xi}_j} m_j | \hat{\imath} \rangle
\tag{52}
$$

$$\mathcal{V}_{fk}(\vec{z}) = -\frac{\hbar}{m_N L^3} \, \mathrm{e}^{\mathrm{i}(\vec{k}-\vec{q})\cdot\vec{z}-\frac{1}{2}(\vec{k}-\vec{q})^2 r_c^2} \langle \hat{f} | \sum_j \mathrm{e}^{-\mathrm{i}(\vec{k}-\vec{q})\cdot\vec{\xi}_j} m_j | \hat{k} \rangle. \tag{53}$$

We now simplify equation (12) by making the approximation that the photon energy $\hbar\omega_p$ is much larger than both the internal energy differences and the center-of-mass recoil energy, that is, that $\hbar\omega_p$ is much larger than $E_i - E_k$ and $E_f - E_k$. With this approximation (which is analogous to the approximation made by Fu and also made in equations (42)–(44) of our hydrogen atom calculation), equation (12) simplifies to

$$E\big[|\mathcal{T}_{fi}^{(2)}|^2\big] = \frac{\gamma t}{\hbar^2} \int \mathrm{d}^3 z \left| \sum_k \frac{\mathcal{V}_{fk}(\vec{z}) \mathcal{W}_{ki}^p - \mathcal{W}_{fk}^p \mathcal{V}_{ki}(\vec{z})}{\hbar\omega_p} \right|^2. \tag{54}$$

Substituting equations (50)–(53) into equation (54), summing over the final state f by the analog of equation (36), and using completeness twice together with algebraic simplification using equation (46), we get for the power radiated

$$\frac{\mathrm{d}P}{\mathrm{d}p} = \frac{2\gamma}{(2\pi)^4} \frac{\hbar}{m_N^2 c^3} \frac{1}{p} \int \frac{\mathrm{d}\Omega_{\hat{p}}}{4\pi} \int \mathrm{d}^3 w \, \mathrm{e}^{-\vec{w}^2 r_c^2} [\vec{w}^2 - (\vec{w}\cdot\hat{p})^2] \langle \hat{\imath} || \mathcal{N}|^2 |\hat{\imath}\rangle$$
$$\mathcal{N} = \sum_j \mathrm{e}^{-\mathrm{i}(\vec{p}-\vec{w})\cdot\vec{\xi}_j} e_j. \tag{55}$$

Note that the internal integration to be used in evaluating the matrix element in this formula includes the Jacobian J of equation (47), and so is

$$|J| \prod_{j=1}^{n-1} \int \mathrm{d}^3 \xi_j. \tag{56}$$

To check that equation (55) reproduces the result of the first term of equation (43) for the hydrogen atom, we note first that for a two-particle system one has $\vec{x}_1 = \vec{X} + \vec{\xi}_1$, $\vec{x}_2 = \vec{X} + \vec{\xi}_2$, and so $\vec{x} = \vec{x}_1 - \vec{x}_2 = \vec{\xi}_1 - \vec{\xi}_2$, which by equation (45) reduces to $\vec{x} = \vec{\xi}_1(1 + m_1/m_2)$. Hence $|J| \mathrm{d}^3 \xi_1 = (1 + m_1/m_2)^3 \mathrm{d}^3 \xi_1 = \mathrm{d}^3 x$, so the internal integration involves the conventional internal coordinate used for the hydrogen atom. The expansion in the small parameter of equation (37) is equivalent, in the many-body context, to setting $\vec{w} = 0$ in $\mathcal{N}$ in equation (55), an approximation that permits the integration over $\vec{w}$ to be easily done, yielding our previous formula for the hydrogen atom radiated power.

One can also apply equation (55) to the case of a crystal lattice. Again making the approximation of neglecting $\vec{w}$ in $\mathcal{N}$, that is, taking r_c to be large, we define

$$f \equiv \sum_{\text{cell}} \mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{\xi}_i} e_i. \tag{57}$$

We then find that the matrix element appearing in equation (55) takes the form (with $\langle \cdots \rangle$ denoting an expectation in the initial state $|\hat{\imath}\rangle$, and with $\vec{L}_i$ a lattice displacement),

$$\langle |\mathcal{N}|^2 \rangle = N_{\text{cell}} \left(\langle |f|^2 \rangle - |\langle f \rangle|^2 \right) + \left| \sum_L \mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{L}_i} \right|^2 |\langle f \rangle|^2$$
$$\cong N_{\text{cell}} \langle |f - \langle f \rangle|^2 \rangle, \tag{58}$$

since the second term on the first line of equation (58) grows more slowly than N_{cell} for generic values of $\vec{p}$. Hence as long as the variance of f over a unit cell is nonzero, the radiated power scales as the size of the crystal lattice (at least for lattice dimensions smaller than r_c).

7. Generalizations and discussion

Several generalizations of the formulas given above can be easily derived. First of all, if the noise Hamiltonian of equation (5) involves general couplings g_i that may differ from the masses m_i, so that

$$H_n = \int d^3z \frac{dW_t(\vec{z})}{dt} \mathcal{V}(\vec{z}, \{x\}), \qquad \mathcal{V}(\vec{z}, \{\vec{x}\}) = -\frac{\hbar}{m_N} \sum_j g_j g(\vec{z} - \vec{x}_j), \tag{59}$$

then in $\mathcal{N}$ in equation (55) one replaces e_j by $e_j g_j / m_j$. Secondly, our calculation, in the non-white-noise case, can be viewed as calculating the radiation produced by a random gravitational potential

$$V_{\text{grav}}(\vec{x}, t) = \sum_i m_i \phi(\vec{x}_i, t), \tag{60}$$

with $\langle \phi \rangle_{AV} = 0$ and with the correlation function

$$\langle \phi(\vec{x}, t) \phi(\vec{x}', t') \rangle_{AV} = \left(\frac{\hbar}{m_N} \right)^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, \gamma(\omega) \, e^{-i\omega(t-t')} G(\vec{x} - \vec{x}'), \tag{61}$$

$$G(\vec{x} - \vec{x}') = \int d^3z g(\vec{x} - \vec{z}) g(\vec{z} - \vec{x}'). \tag{62}$$

Since for the Gaussian g of equation (6) one has

$$\int d^3x \, e^{i\vec{k}\cdot\vec{x}} g(\vec{x}) = e^{-\frac{1}{2}\vec{k}^2 r_c^2}, \tag{63}$$

$$\int d^3x \, e^{i\vec{k}\cdot\vec{x}} G(\vec{x}) = \int d^3x \, e^{i\vec{k}\cdot\vec{x}} \int d^3y \, g(\vec{x} - \vec{y}) g(\vec{y}) = e^{-\vec{k}^2 r_c^2}, \tag{64}$$

for a general $G(\vec{x})$ in equation (61) one simply replaces $e^{-\vec{w}^2 r_c^2}$ in the radiated power expressions by

$$G[\vec{w}] = \int d^3x \, e^{i\vec{w}\cdot\vec{x}} G(\vec{x}). \tag{65}$$

Finally, for a more general non-white noise that does not have a time-translation invariant correlation function, so that equation (8) is replaced by

$$E\left[\frac{dW_t(\vec{x})}{dt} \frac{dW_{t'}(\vec{y})}{dt'} \right] = \Delta(t, t')\delta^3(\vec{x} - \vec{y}), \tag{66}$$

the master formula in the non-white-noise case is modified by replacing

$$t\gamma \left(\omega_p + \frac{E_f - E_i}{\hbar} \right) \tag{67}$$

by

$$\int_0^t ds \int_0^t du \, \Delta(s, u) \, e^{i(s-u)[\omega_p + \frac{E_f - E_i}{\hbar}]}. \tag{68}$$

The most general case, in which the correlation function of equation (61) does not factorize into a temporal correlation times a spatial correlation, can be obtained by combining results from equations (61)–(68).

To conclude, we consider the implications of our results for CSL model phenomenology. Since we have seen that for a hydrogenic or a general atomic system emitting kilovolt gamma

rays, charge neutrality does not imply a corresponding cancellation in the radiation rate, the estimates of Fu [3] must be taken as giving the best upper bounds on the CSL parameter λ (defined following equation (20)) for the white-noise case. Including [2] a factor of 4π correction to Fu's evaluation of the electric charge squared e^2, as well as [8] a factor of roughly 4 increase in the experimental rate limit subsequent to the value used by Fu, Fu's calculation implies the bound $\lambda < 7 \times 10^{-11}$ s^{-1}, which is $\sim 3 \times 10^6$ larger than the standard CSL model value of $\lambda = 2.2 \times 10^{-17}$ s^{-1}. As we noted in section 1, this upper bound is several orders of magnitude below the lower bound on λ set by postulating that latent image formation (as opposed to image development) should correspond to state vector reduction. Although increasing r_c to 10^{-4} cm decreases the 11 kV photon radiation rate, and so increases the corresponding upper bound on λ, by two orders of magnitude, as discussed in [2] this increase in r_c also increases the latent image formation lower bound on λ by one to two orders of magnitude, and so does not eliminate the potential discrepancy.

By contrast, in the non-white-noise case there is not necessarily a conflict, since the relevant radiation rate involves the noise spectral coefficient $\gamma(\omega)$ at a frequency of at least that of the emitted gamma ray, of order 10^{18} s^{-1}. In fact, in their review [1], Bassi and Ghirardi suggest a cutoff in the noise frequency spectrum of order $c/r_c \sim 10^{15}$ s^{-1}, which would be more than sufficient. Even a much lower frequency cutoff would suffice to explain reduction in typical measurements with measurement times of order a nanosecond or longer; for example, a cutoff of order 10^{11} s^{-1} would be more than adequate. This would correspond to an energy cutoff of order 10^{-4} eV, or a noise temperature of order 1 degree K. So possibly even a non-white cosmic relic background noise field, with suitable correlator structure, coupling as a real-valued noise term $\mathcal{N}$ in the Schrödinger equation for dψ, could explain state vector reduction in measurement situations, without coming close to violating the upper bound set by Fu's calculation.

Acknowledgments

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